

# TOPOLOGICAL INVARIANTS AND CORNER STATES FOR HAMILTONIANS ON A THREE DIMENSIONAL LATTICE

SHIN HAYASHI

ABSTRACT. Periodic Hamiltonians on a three dimensional lattice which have a spectral gap not only on the bulk but also on two edges at the common Fermi level are considered. By using  $K$ -theory applied for quarter-plane Toeplitz algebras, two topological invariants are defined for such gapped Hamiltonians. One is defined for the bulk and edges, and the other corresponds to wave functions localized near the corner. A correspondence of these two invariants is proved.

## CONTENTS

1. Introduction	1
2. Preliminaries	3
2.1. $K$ -theory for $C^*$ -algebras	3
2.2. Spectral Flow and Winding Number	4
2.3. Quarter-Plane Toeplitz $C^*$ -algebras	5
3. Bulk-Edge and Corner Correspondence	7
3.1. Bulk-Edge Invariant	7
3.2. Corner Invariant	8
3.3. Correspondence	8
References	9

## 1. INTRODUCTION

In condensed matter physics, it is known a correspondence between topological invariants defined for a gapped Hamiltonian of an infinite system without edge and that for a Hamiltonian of a system with edge. This correspondence is called the bulk-edge correspondence. In the theoretical study of the quantum Hall effect, an integer valued topological invariant for a gapped Hamiltonian of an infinite system without edge was introduced by D. J. Thouless, M. Kohmoto, M. P. Nightingale and M. den Nijs [TKNdN82]. This invariant is called the TKNN number, which is the first Chern number of the Bloch bundle [Koh85]. Y. Hatsugai considered such phenomena on a system with edge, and defined a topological invariant as a winding number counted on a Riemann surface, in other words, the spectral flow of a family of self-adjoint Fredholm Toeplitz operators [Hat93a]. The equality between

---

*Date:* January 17, 2017.

*2010 Mathematics Subject Classification.* Primary 19K56; Secondary 47B35, 81V99.

*Key words and phrases.* bulk-edge invariants, bulk-edge and corner correspondence, quarter-plane Toeplitz operators,  $K$ -theory for  $C^*$ -algebras.

these two topological invariants (bulk-edge correspondence) was proved by Hatsugai [Hat93b].

Since the work of J. Bellissard [Bel86, BvESB94],  $K$ -theory for  $C^*$ -algebras is known to be very useful in this context. J. Kellendonk, T. Richter and H. Schulz-Baldes proved the bulk-edge correspondence by using the six-term exact sequence of  $K$ -Theory for  $C^*$ -algebras associated to the following Toeplitz extension,

$$0 \rightarrow K(l^2(\mathbb{Z})) \rightarrow \mathcal{T} \rightarrow C(\mathbb{T}) \rightarrow 0,$$

where  $K(l^2(\mathbb{Z}))$  is the  $C^*$ -algebra of compact operators on  $l^2(\mathbb{Z})$ ,  $\mathcal{T}$  is the Toeplitz algebra and  $C(\mathbb{T})$  is the  $C^*$ -algebra consists of continuous functions on the unit circle  $\mathbb{T}$  in the complex plane with uniform norm [SBKR00, KRSB02]. There are many mathematical works (see a comprehensive account [PSB16] and references therein).

Apart from the study of topological phases, the analysis of Toeplitz operators was developed (see [Dou98, BS06], for example). We here focus on the theory of quarter-plane Toeplitz operators. Such operators were first studied by R. G. Douglas and R. Howe, and many results were obtained [DH71, CDSS71, CDS72]. Among other things, E. Park showed that there is a short exact sequence for  $C^*$ -algebras,

$$(1) \quad 0 \rightarrow \mathcal{K} \rightarrow \mathcal{T}^{\alpha, \beta} \rightarrow \mathcal{S}^{\alpha, \beta} \rightarrow 0.$$

where  $\mathcal{T}^{\alpha, \beta}$  is the quarter-plane Toeplitz algebra (the meaning of each symbols are explained in Sect. 2.3). Park studied quarter-plane Toeplitz algebras by using  $K$ -theory for  $C^*$ -algebras [Par90, PS91].

In this paper, we mainly consider a system with corner which appears as the intersection of two edges. We consider a periodic Hamiltonian on the three dimensional lattice  $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ , which has a spectral gap not only on the bulk (which means that our Hamiltonian is gapped) but also on two edges (which means that restrictions of our Hamiltonian onto two semigroups of our lattice which correspond to two edges, are gapped) at the common Fermi level. For such systems, we define two topological invariants. One is defined for the bulk and edges, and the other is defined for the corner. Both invariants are defined as elements of some  $K$ -groups. In our settings, topological invariants considered in the case of the bulk-edge correspondence are zero since our edges are also gapped. In this sense, invariants considered in this paper can be seen as secondary invariants. We next show a correspondence of these two invariants, by using the six-term exact sequence associated to (1). We have maps from these  $K$ -groups to  $\mathbb{Z}$ , and our invariants corresponds to a spectral flow of a one-parameter family of self-adjoint Fredholm quarter-plane Toeplitz operators. Although we mainly consider some three dimensional systems, our method can also be applied to some systems of other dimensions (Remark 3.10)

Note that, except for the use of the sequence (1), our invariants are defined and the correspondence is proved as in the case of the bulk-edge correspondence. Although our study of such invariants are modeled on Kellendonk–Richter–Schulz-Baldes’ proof of the bulk-edge correspondence, we note that there are many other proofs of the bulk-edge correspondence [EG02, ASBVB13, GP13, BCE15, Hay16, Kub16, MT16].

There also is a bulk-edge correspondence for systems with symmetry (time-reversal symmetry for quantum spin Hall systems, for example) [ASBVB13, GP13, Kub16, MT16, BKR17]. So it is natural to expect such a “bulk-edge and corner” correspondence for systems with symmetries, which is not treated in this paper.

This paper is organized as follows. In Sect. 2, some basic facts about  $K$ -theory for  $C^*$ -algebras, spectral flow and quarter-plane Toeplitz operators, which are used in this paper, is collected. In Sect. 3, conditions for “gapped” Hamiltonians which are considered in this paper are fixed. Two topological invariants for such Hamiltonians are defined, and the correspondence of these two is proved.

*Acknowledgement.* The author expresses gratitude to his supervisor Mikio Furuta for his support and encouragement. This work is motivated by author’s collaborative research with Mikio Furuta, Motoko Kotani, Yosuke Kubota, Shinichiroh Matsuo and Koji Sato. He would like to thank them for many stimulating conversations and encouragements.

## 2. PRELIMINARIES

In this section, we collect some basic facts and calculations needed in this paper. Throughout this paper, all algebras and Hilbert spaces are considered over the complex field  $\mathbb{C}$ , and all operators are complex linear.

**2.1.  $K$ -theory for  $C^*$ -algebras.** In this paper, we use  $K$ -theory for  $C^*$ -algebras in order to define some topological invariants and also to prove our main theorem. This subsection collects some basic facts from  $K$ -theory for  $C^*$ -algebras without proof. We refer the reader to [Mur90, WO93, Bla98, HR00, RLL00] for the details.

Let  $A$  be a unital  $C^*$ -algebra, that is a Banach  $*$ -algebra with multiplicative unit which satisfies  $\|a^*a\| = \|a\|^2$  for any  $a$  in  $A$ . An element  $p$  in  $A$  is called a projection if  $p = p^* = p^2$ , and an element  $u$  in  $A$  is said to be unitary if  $u^*u = uu^* = 1$ . For a positive integer  $n$ , let  $M_n(A)$  be the matrix algebra of all  $n \times n$  matrices with entries in  $A$ . As in the case of  $M_n(\mathbb{C})$ , the matrix algebra  $M_n(A)$  has a natural  $*$ -algebra structure. We know that  $M_n(A)$  has a (unique) norm making it a  $C^*$ -algebra. We denote the set of all projections in  $M_n(A)$  and let  $P_\infty(A) := \bigsqcup_{n=1}^\infty P_n(A)$  by  $P_n(A)$ . For  $p \in P_n(A)$  and  $q \in P_m(A)$ , we write  $p \sim_0 q$  if and only if there exists some  $v \in M_{m,n}(A)$  such that  $p = v^*v$  and  $q = vv^*$ . The relation  $\sim_0$  is an equivalence relation on  $P_\infty(A)$ . We define a binary operation  $\oplus$  on  $P_\infty(A)$  by  $p \oplus q := \text{diag}(p, q)$ . Then  $\oplus$  induces an addition  $+$  on equivalence classes  $D(A) := P_\infty(A)/\sim_0$ , and  $(D(A), +)$  is a commutative monoid. The  $K_0$ -group  $K_0(A)$  for a unital  $C^*$ -algebra  $A$  is defined to be the group completion (Grothendieck group) of the commutative monoid  $(D(A), +)$ . We denote the class of  $p \in P_\infty(A)$  in  $K_0(A)$  by  $[p]_0$ . For a non-unital  $C^*$ -algebra  $I$ , we define its  $K_0$ -group  $K_0(I)$  to be the kernel of the map  $K_0(\tilde{I}) \rightarrow K_0(\mathbb{C})$ , where  $\tilde{I} = I \oplus \mathbb{C}$  is the unitization of  $I$ , and the map is induced by the projection onto the second component. Let  $\mathcal{U}(A)$  be the group of unitary elements in  $A$ , and let  $\mathcal{U}_n(A) := \mathcal{U}(M_n(A))$ . We consider the set  $\mathcal{U}_\infty(A) := \bigsqcup_{n=1}^\infty \mathcal{U}_n(A)$ . Let  $\oplus$  be a binary operation on  $\mathcal{U}_\infty(A)$  defined as above. For  $u \in \mathcal{U}_n(A)$  and  $v \in \mathcal{U}_m(A)$ , we write  $u \sim_1 v$  if and only if there exists some  $k \geq \max\{m, n\}$  such that  $u \oplus 1_{k-n}$  and  $v \oplus 1_{k-m}$  are homotopic in  $\mathcal{U}_k(A)$ . The relation  $\sim_1$  is an equivalence relation on  $\mathcal{U}_\infty(A)$ , and  $\oplus$  induces an addition  $+$  on equivalence classes  $\mathcal{U}_\infty(A)/\sim_1$ . Then  $(\mathcal{U}_\infty(A)/\sim_1, +)$  is an abelian group. We denote this group by  $K_1(A)$  and the class of  $u \in \mathcal{U}_\infty(A)$  in  $K_1(A)$  by  $[u]_1$ . For a non-unital  $C^*$ -algebra  $I$ , its  $K_1$ -group is defined by using its unitization,  $K_1(I) := K_1(\tilde{I})$ . Note that, by using the polar decomposition, an invertible element in  $M_n(A)$  defines an element in  $K_1(A)$ .  $*$ -homomorphisms  $\varphi, \psi: A \rightarrow B$  between  $C^*$ -algebras are said to be homotopic if there exists a path of  $*$ -homomorphisms

$\varphi_t: A \rightarrow B$  for  $t \in [0, 1]$  such that the map  $[0, 1] \rightarrow B$  defined by  $t \mapsto \varphi_t(a)$  is continuous for each  $a \in A$ ,  $\varphi_0 = \varphi$  and  $\varphi_1 = \psi$ . In this sense,  $K_0$  and  $K_1$  are the additive covariant homotopy functor from the category of  $C^*$ -algebras to the category of abelian groups. Let  $H$  be a separable Hilbert space and  $K(H)$  be the set of compact operators on  $H$ . For a  $C^*$ -algebra  $A$ , we have the stability property, that is,  $K_i(K(H) \otimes A) \cong K_i(A)$  where  $i = 0, 1$  and  $\otimes$  is the  $C^*$ -algebraic tensor product.

Let  $A$  be a  $C^*$ -algebra. The suspension of  $A$  is the  $C^*$ -algebra  $SA = \{f \in C([0, 1], A) \mid f(0) = f(1) = 0\} \cong A \otimes C_0((0, 1))$ , where  $C([0, 1], A)$  is the  $C^*$ -algebra of continuous functions from  $[0, 1]$  to  $A$ , and  $C_0((0, 1))$  is the  $C^*$ -algebra of complex valued continuous functions which vanish at infinity. Then there is an isomorphism  $\theta_A: K_1(A) \rightarrow K_0(SA)$ . We also have a map  $\beta_A: K_0(A) \rightarrow K_1(SA)$ , called the Bott map. If  $A$  is a unital  $C^*$ -algebra,  $\beta_A$  is given by  $\beta_A[p]_0 = [f_p]_1$  where  $f_p(t) = \exp(2\pi i t p)$ . By the Bott periodicity theorem,  $\beta_A$  is an isomorphism. For a short exact sequence of  $C^*$ -algebras  $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ , there associates the following six-term exact sequence.

$$\begin{array}{ccccc} K_1(I) & \longrightarrow & K_1(A) & \longrightarrow & K_1(B) \\ \delta_0 \uparrow & & & & \downarrow \delta_1 \\ K_0(B) & \longleftarrow & K_0(A) & \longleftarrow & K_0(I). \end{array}$$

The map  $\delta_0$  is called the exponential map. If  $A$  and  $B$  are unital  $C^*$ -algebras, and  $I$  is a closed ideal in  $A$ , then  $\delta_0$  is expressed in the following way. For  $p \in P_n(B)$ , we can take its self-adjoint lift  $\hat{p} \in M_n(A)$ , and then we have  $\delta_0[p]_0 = [\exp(-2\pi i \hat{p})]_1$ . Note that the following diagram is commutative.

$$\begin{array}{ccc} K_1(B) & \xrightarrow{\theta_B} & K_0(SB) \\ \delta_1 \downarrow & & \downarrow \delta_0 \\ K_0(I) & \xrightarrow{\beta_I} & K_1(SI). \end{array}$$

**2.2. Spectral Flow and Winding Number.** In this subsection, we discuss some relationship between the spectral flow and the winding number [AS69, APS75, Phi96].

Let  $\mathbb{R}_{>0}$  (resp.  $\mathbb{R}_{<0}$ ) be the set of strictly positive (resp. negative) real numbers. For a separable Hilbert space  $H$ , we consider the space of bounded linear operators  $B(H)$  on  $H$  with norm topology. Let  $\mathcal{F}_*^{s.a.}$  be the subspace of  $B(H)$  consists of all self-adjoint Fredholm operators whose essential spectrum<sup>1</sup> does not contained neither  $\mathbb{R}_{>0}$  nor  $\mathbb{R}_{<0}$ . We consider the following subspace of  $\mathcal{F}_*^{s.a.}$ ,

$$\hat{\mathcal{F}}_*^\infty := \{B \in \mathcal{F}_*^{s.a.} \mid \|B\| = 1, \text{ sp}(B) \text{ is finite and } \text{ess-sp}(B) = \{\pm 1\}\}.$$

Let  $i: \hat{\mathcal{F}}_*^\infty \rightarrow \mathcal{F}_*^{s.a.}$  be the inclusion. Then  $i$  is a homotopy equivalence. Let  $\mathbf{U}(\infty)$  be the inductive limit of a sequence  $\mathbf{U}(1) \rightarrow \mathbf{U}(2) \rightarrow \cdots$ , where  $\mathbf{U}(n)$  is the unitary group of degree  $n$ , and the map  $\mathbf{U}(n) \rightarrow \mathbf{U}(n+1)$  is given by  $\mathbf{A} \mapsto \text{diag}(\mathbf{A}, 1)$ .

<sup>1</sup>In this paper, we denote the spectrum of an operator  $T$  by  $\text{sp}(T)$ , and the essential spectrum of  $T$  by  $\text{ess-sp}(T)$ . For a self-adjoint operator  $T$ , the essential spectrum of  $T$  consists of accumulation points of  $\text{sp}(T)$  and isolated points of  $\text{sp}(T)$  with infinite multiplicity (see Proposition 2.2.2 of [HR00], for example).

We have a map  $j: \hat{F}_*^\infty \rightarrow \mathbf{U}(\infty)$  given by  $j(B) = \exp(i\pi(B + 1))$ . The map  $j$  also is a homotopy equivalence.<sup>2</sup> Thus we have  $K_1(C(\mathbb{T})) = [\mathbb{T}, \mathbf{U}(\infty)] \cong [\mathbb{T}, \hat{F}_*^\infty] \cong [\mathbb{T}, \mathcal{F}_*^{s.a.}]$ . For a continuous loop in  $\mathcal{F}_*^{s.a.}$ , its spectral flow is defined. This is, roughly speaking, the net number of crossing points of eigenvalues with zero counted with multiplicity. Then the following is commutative diagram.

$$(2) \quad \begin{array}{ccc} [\mathbb{T}, \mathcal{F}_*^{s.a.}] & \xrightarrow{\text{sf}} & \mathbb{Z} \\ i_* \uparrow & & \uparrow \\ [\mathbb{T}, \hat{F}_*^\infty] & \xrightarrow{j_*} & [\mathbb{T}, \mathbf{U}(\infty)], \end{array}$$

where the map  $\text{sf}: [\mathbb{T}, \mathcal{F}_*^{s.a.}] \rightarrow \mathbb{Z}$  is given by the spectral flow, and the map  $[\mathbb{T}, \mathbf{U}(\infty)] \rightarrow \mathbb{Z}$  is given by taking the winding number of the determinant. All arrows are group isomorphisms.

**2.3. Quarter-Plane Toeplitz  $C^*$ -algebras.** In this subsection, we collect basic facts about quarter-plane Toeplitz operators which are used later.

Let  $\mathcal{H}$  be the Hilbert space  $l^2(\mathbb{Z} \times \mathbb{Z})$ . For a pair of integers  $(m, n)$ , let  $e_{m,n}$  be the element of  $\mathcal{H}$  that is 1 at  $(m, n)$  and 0 elsewhere. For such  $(m, n)$ , let  $M_{m,n}: \mathcal{H} \rightarrow \mathcal{H}$  be the translation operator defined by  $(M_{m,n}\varphi)(k, l) = \varphi(m+k, n+l)$ . We choose real numbers  $\alpha < \beta$ , and let  $\mathcal{H}^\alpha$  and  $\mathcal{H}^\beta$  be closed subspaces of  $\mathcal{H}$  spanned by  $\{e_{m,n} \mid -\alpha m + n \geq 0\}$  and  $\{e_{m,n} \mid -\beta m + n \geq 0\}$ , respectively. Let  $\mathcal{H}^{\alpha,\beta}$  be their intersection  $\mathcal{H}^\alpha \cap \mathcal{H}^\beta$  (see Figure 1.). We can take  $\alpha = -\infty$  or  $\beta = +\infty$ ,

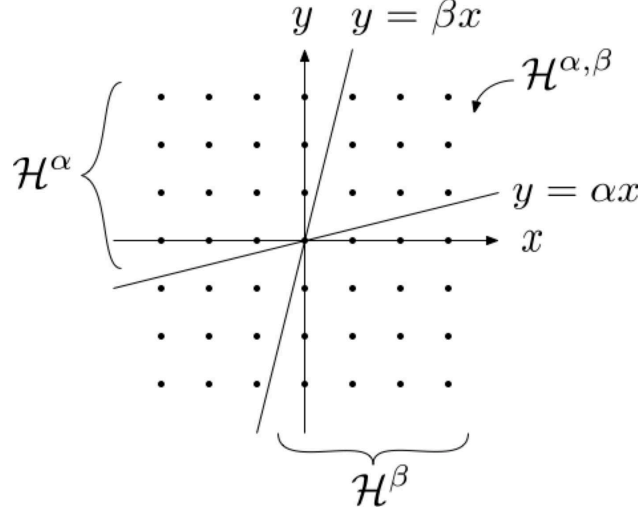


FIGURE 1. Half planes  $\mathcal{H}^\alpha$  and  $\mathcal{H}^\beta$ , and the quarter-plane  $\mathcal{H}^{\alpha,\beta}$  in  $\mathbb{Z} \times \mathbb{Z}$

but not both. Let  $P^\alpha$  and  $P^\beta$  be orthogonal projections of  $\mathcal{H}$  onto  $\mathcal{H}^\alpha$  and  $\mathcal{H}^\beta$ ,

<sup>2</sup>The subspace  $\hat{F}_*^\infty$  and the commutativity of the diagram (2) was discussed explicitly in [Phi96], while the homotopy equivalence between  $\hat{F}_*^\infty$  and  $\mathbf{U}(\infty)$  was essentially proved in [AS69]. We here follow the notations used in [Phi96].

respectively. Then  $P^\alpha P^\beta$  be the orthogonal projection of  $\mathcal{H}$  onto  $\mathcal{H}^{\alpha,\beta}$ . We define the *quarter-plane Toeplitz  $C^*$ -algebra* to be the  $C^*$ -algebra  $\mathcal{T}^{\alpha,\beta}$  generated by  $\{P^\alpha P^\beta M_{m,n} P^\alpha P^\beta \mid (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$ . We also define *half-plane Toeplitz  $C^*$ -algebras*  $\mathcal{T}^\alpha$  and  $\mathcal{T}^\beta$  to be  $C^*$ -algebras generated by  $\{P^\alpha M_{m,n} P^\alpha \mid (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$  and  $\{P^\beta M_{m,n} P^\beta \mid (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$ , respectively. We have surjective  $*$ -homomorphisms  $\gamma^\alpha: \mathcal{T}^{\alpha,\beta} \rightarrow \mathcal{T}^\alpha$  and  $\gamma^\beta: \mathcal{T}^{\alpha,\beta} \rightarrow \mathcal{T}^\beta$ , which map  $P^\alpha P^\beta M_{m,n} P^\alpha P^\beta$  to  $P^\alpha M_{m,n} P^\alpha$  and  $P^\beta M_{m,n} P^\beta$ , respectively. We also have surjective  $*$ -homomorphisms  $\sigma^\alpha: \mathcal{T}^\alpha \rightarrow C(\mathbb{T} \times \mathbb{T})$  and  $\sigma^\beta: \mathcal{T}^\beta \rightarrow C(\mathbb{T} \times \mathbb{T})$  which map  $P^\alpha M_{m,n} P^\alpha$  to  $\chi_{m,n}$  and  $P^\beta M_{m,n} P^\beta$  to  $\chi_{m,n}$ , respectively, where  $\chi_{m,n}(z_1, z_2) = z_1^m z_2^n$ . Well-definedness of  $\gamma^\alpha$  and  $\gamma^\beta$  is proved in [Par90], and that of  $\sigma^\alpha$  and  $\sigma^\beta$  is proved in [CD71]. Let  $\rho^\alpha: \mathcal{T}^\alpha \rightarrow \mathcal{T}^{\alpha,\beta}$ ,  $\rho^\beta: \mathcal{T}^\beta \rightarrow \mathcal{T}^{\alpha,\beta}$  and  $\xi^{\alpha,\beta}: C(\mathbb{T} \times \mathbb{T}) \rightarrow \mathcal{T}^{\alpha,\beta}$  be bounded linear maps given by compression, that is,  $\rho^\alpha(X) = P^\beta X P^\beta$ ,  $\rho^\beta(Y) = P^\alpha Y P^\alpha$  and  $\xi^{\alpha,\beta}(Z) = P^\alpha P^\beta Z P^\alpha P^\beta$ , respectively. We define a  $C^*$ -algebra  $\mathcal{S}^{\alpha,\beta}$  to be the pullback of  $\mathcal{T}^\alpha$  and  $\mathcal{T}^\beta$  along  $C(\mathbb{T} \times \mathbb{T})$ , that is,  $\mathcal{S}^{\alpha,\beta} := \{(T^\alpha, T^\beta) \in \mathcal{T}^\alpha \oplus \mathcal{T}^\beta \mid \sigma^\alpha(T^\alpha) = \sigma^\beta(T^\beta)\}$ . There is a  $*$ -homomorphism  $\gamma: \mathcal{T}^{\alpha,\beta} \rightarrow \mathcal{S}^{\alpha,\beta}$  given by  $\gamma(T) = (\gamma^\alpha(T), \gamma^\beta(T))$ . Let  $\mathcal{K}$  be the  $C^*$ -algebra of compact operators on  $\mathcal{H}^{\alpha,\beta}$ .

**Theorem 2.1** (Park[Par90]). *There is the following short exact sequence,*

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T}^{\alpha,\beta} \xrightarrow{\gamma} \mathcal{S}^{\alpha,\beta} \rightarrow 0,$$

*which has a linear splitting  $\rho: \mathcal{S}^{\alpha,\beta} \rightarrow \mathcal{T}^{\alpha,\beta}$  given by  $\rho(T^\alpha, T^\beta) = \rho^\alpha(T^\alpha) + \rho^\beta(T^\beta) - \xi^{\alpha,\beta}(\sigma^\alpha(T^\alpha), \sigma^\beta(T^\beta))$ .*

The following theorem follows immediately by using Atkinson's theorem.<sup>3</sup>

**Theorem 2.2** (Douglas-Howe[DH71], Park[Par90]). *An operator  $T$  in  $\mathcal{T}^{\alpha,\beta}$  is a Fredholm operator if and only if  $\gamma^\alpha(T)$  and  $\gamma^\beta(T)$  are both invertible elements in  $\mathcal{T}^\alpha$  and  $\mathcal{T}^\beta$ , respectively.*

By taking a tensor product of the above sequence and  $C(\mathbb{T})$ , we have the short exact sequence,<sup>4</sup>  $0 \rightarrow \mathcal{K} \otimes C(\mathbb{T}) \rightarrow \mathcal{T}^{\alpha,\beta} \otimes C(\mathbb{T}) \rightarrow \mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T}) \rightarrow 0$ . Associated to this sequence, we have the following six-term exact sequence.

$$\begin{array}{ccccc} K_1(\mathcal{K} \otimes C(\mathbb{T})) & \longrightarrow & K_1(\mathcal{T}^{\alpha,\beta} \otimes C(\mathbb{T})) & \longrightarrow & K_1(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T})) \\ \delta_0 \uparrow & & & & \downarrow \delta_1 \\ K_0(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T})) & \longleftarrow & K_0(\mathcal{T}^{\alpha,\beta} \otimes C(\mathbb{T})) & \longleftarrow & K_0(\mathcal{K} \otimes C(\mathbb{T})). \end{array}$$

Note that  $K_1(\mathcal{K} \otimes C(\mathbb{T}))$  is isomorphic to  $K_1(C(\mathbb{T}))$ .

**Lemma 2.3.** *The map  $\delta_0: K_0(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T})) \rightarrow K_1(C(\mathbb{T}))$  is surjective.*

*Proof.* Let us take a base point of  $\mathbb{T}$ , then we have isomorphisms<sup>5</sup>  $K_0(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T})) \cong K_0(\mathcal{S}^{\alpha,\beta}) \oplus K_1(\mathcal{S}^{\alpha,\beta})$  and  $K_1(\mathcal{K} \otimes C(\mathbb{T})) \cong K_1(\mathcal{K}) \oplus K_0(\mathcal{K}) \cong 0 \oplus \mathbb{Z}$ .

<sup>3</sup>Such a necessary and sufficient condition was first obtained by Douglas-Howe [DH71] for the special case  $\mathcal{T}^{0,\infty}$  by expressing the algebra  $\mathcal{T}^{0,\infty}$  as a tensor product of two Toeplitz algebras. Park obtained such conditions for the general  $\mathcal{T}^{\alpha,\beta}$  in a different way [Par90].

<sup>4</sup>Since  $C(\mathbb{T})$  is an abelian  $C^*$ -algebra,  $C(\mathbb{T})$  is a nuclear  $C^*$ -algebra by Takesaki's theorem. Since  $C(\mathbb{T})$  is nuclear, this sequence is exact (see [Mur90], for example).

<sup>5</sup>See Example 7.5.1 of [Mur90], for example.

Consider the following commutative diagram,

$$\begin{array}{ccc} K_0(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T})) & \xrightarrow{\cong} & K_0(\mathcal{S}^{\alpha,\beta}) \oplus K_1(\mathcal{S}^{\alpha,\beta}) \\ \downarrow \delta_0 & & \downarrow \delta_0 \oplus \delta_1 \\ K_1(\mathcal{K} \otimes C(\mathbb{T})) & \xrightarrow{\cong} & 0 \oplus \mathbb{Z}. \end{array}$$

The group  $K_1(\mathcal{S}^{\alpha,\beta})$  is isomorphic to  $\mathbb{Z}$  [Par90], and the map  $\delta_1: K_1(\mathcal{S}^{\alpha,\beta}) \rightarrow K_0(\mathcal{K})$  is an isomorphism [Par90, Jia95]. Thus the left map is surjective.  $\square$

Remark 2.4. Note that the group  $K_0(\mathcal{S}^{\alpha,\beta})$  is calculated as follows [Par90].

$$K_0(\mathcal{S}^{\alpha,\beta}) \cong \begin{cases} \mathbb{Z} & \text{if } \alpha \text{ and } \beta \text{ both rational,} \\ \mathbb{Z}^2 & \text{if one of } \alpha \text{ and } \beta \text{ is rational and the other is irrational,} \\ \mathbb{Z}^3 & \text{if } \alpha \text{ and } \beta \text{ both irrational.} \end{cases}$$

In this sense,  $K_0(\mathcal{S}^{\alpha,\beta})$  depends delicately on angles  $\alpha, \beta$  of edges. By the proof of Lemma 2.3, this component maps to zero by  $\delta_0$ .

### 3. BULK-EDGE AND CORNER CORRESPONDENCE

In this section, we consider some “gapped” Hamiltonians, and define two topological invariants for them. A correspondence between these two is proved.

**3.1. Bulk-Edge Invariant.** Let  $V$  be a finite dimensional Hermitian vector space and denote the complex dimension of  $V$  by  $N$ . Let  $\mathcal{H}_V := \mathcal{H} \otimes V$ ,  $\mathcal{H}_V^\alpha := \mathcal{H}^\alpha \otimes V$ ,  $\mathcal{H}_V^\beta := \mathcal{H}^\beta \otimes V$ ,  $\mathcal{H}_V^{\alpha,\beta} := \mathcal{H}^{\alpha,\beta} \otimes V$ ,  $P_V^\alpha := P^\alpha \otimes 1$  and  $P_V^\beta := P^\beta \otimes 1$ . We consider a continuous family of bounded linear operators,  $H: \mathbb{T} \rightarrow B(L^2(\mathbb{T} \times \mathbb{T}; V))$ , where, for each  $t$  in  $\mathbb{T}$ , the operator  $H(t)$  is a self-adjoint multiplication operator generated by a continuous map.<sup>6</sup> We call  $H(t)$  *bulk Hamiltonians*. By using the Fourier transform  $L^2(\mathbb{T} \times \mathbb{T}; V) \cong \mathcal{H}_V$ , we have a continuous family of self-adjoint bounded linear operators. We write these operators  $H(t): \mathcal{H}_V \rightarrow \mathcal{H}_V$  by using the same symbol.

Example 3.1. We assume that we are given, for each  $(p, q, r) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ , an endomorphism  $A_{p,q,r}$  on  $V$ , which satisfies  $\sum_{(p,q,r) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}} \|A_{p,q,r}\|_{\text{op}} < +\infty$ . We consider an operator,  $H_1: l^2(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}; V) \rightarrow l^2(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}; V)$  defined by,

$$(H_1\varphi)_{(k,l,m)} = \sum_{(p,q,r) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}} A_{p,q,r} \varphi_{k-p, l-q, m-r}.$$

We assume that  $H_1$  is a self-adjoint operator. Then its partial Fourier transform gives an example of our family.

We consider following half-plane Toeplitz operators,

$$H^\alpha(t) := P_V^\alpha H(t) P_V^\alpha: \mathcal{H}_V^\alpha \rightarrow \mathcal{H}_V^\alpha, \quad H^\beta(t) := P_V^\beta H(t) P_V^\beta: \mathcal{H}_V^\beta \rightarrow \mathcal{H}_V^\beta.$$

They are bounded self-adjoint operators. We call  $H^\alpha(t)$  and  $H^\beta(t)$  *edge Hamiltonians*. We take an orthonormal frame of  $V$ . Then, since  $H^\alpha(t)$  and  $H^\beta(t)$  are compression of the same operator  $H(t)$ , the pair  $(H^\alpha(t), H^\beta(t))$  defines a self-adjoint element of the  $C^*$ -algebra  $M_N(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T}))$ . We now take a real number  $\mu$  such

<sup>6</sup>Let  $f: \mathbb{T} \times \mathbb{T} \rightarrow \text{End}(V)$  be a continuous map. Then the operator on  $L^2(\mathbb{T} \times \mathbb{T}; V)$  defined by  $g \mapsto fg$  is called the multiplication operator generated by  $f$ .

that  $\mu$  does not contained neither  $\text{sp}(H^\alpha(t))$  nor  $\text{sp}(H^\beta(t))$  for any  $t$  in  $\mathbb{T}$ . Then the element  $(H^\alpha(t) - \mu, H^\beta(t) - \mu)$  is invertible in  $M_N(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T}))$ .

**Remark 3.2.** Such  $\mu$  does exists. Actually we can take  $\mu$  sufficiently large or small. However, if we choose such  $\mu$ , our topological invariants are zero (see also Remark 3.9). Non-trivial invariants appear if operators  $H^\alpha(t)$  and  $H^\beta(t)$  have a common spectral gap at the Fermi level  $\mu$ . Note that, in this case, our Hamiltonian  $H(t)$  also has a spectral gap at  $\mu$  since  $\text{sp}(H(t))$  is contained in  $\text{sp}(H^\alpha(t), H^\beta(t))$ .

By Remark 3.2, we further assume that  $\text{sp}(H^\alpha(t) - \mu, H^\beta(t) - \mu)$  does not contained neither  $\mathbb{R}_{>0}$  nor  $\mathbb{R}_{<0}$ . We refer to our assumption about the choice of  $\mu$  as the *spectral gap condition*.

We next define some topological invariants for our operators. We consider following subspaces of the complex plane.

$$\Pi^+ := \{z \in \mathbb{C} \mid \text{Re}(z) > 0\}, \quad \Pi^- := \{z \in \mathbb{C} \mid \text{Re}(z) < 0\}, \quad \Pi := \Pi^+ \sqcup \Pi^-.$$

Let  $h: \Pi \rightarrow \mathbb{C}$  be a continuous function which is 0 on  $\Pi^+$  and 1 on  $\Pi^-$ .

**Definition 3.3.** By the continuous functional calculus, we have a projection  $p := h(H^\alpha(t) - \mu, H^\beta(t) - \mu)$  in  $M_N(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T}))$ . We denote the element  $[p]_0$  in the  $K$ -group  $K_0(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T}))$  by  $\mathcal{I}_{\text{Bulk-Edge}}$ , and call the *bulk-edge invariant*.<sup>7</sup>

**Remark 3.4.** Note that, in order to define the bulk-edge invariant, we use just the information of “bulk and edge”, and do not use the information of “corner”. This is a justification of the name of the “bulk-edge invariant”.

**3.2. Corner Invariant.** We consider following quarter-plane Toeplitz operators,

$$H^{\alpha,\beta}(t) := P_V^\alpha P_V^\beta H(t) P_V^\alpha P_V^\beta: \mathcal{H}_V^{\alpha,\beta} \rightarrow \mathcal{H}_V^{\alpha,\beta}.$$

We call  $H^{\alpha,\beta}(t)$  *corner Hamiltonians*.

**Definition 3.5.** By Theorem 2.2, we have a continuous family  $\{H^{\alpha,\beta}(t) - \mu\}_{t \in \mathbb{T}}$  of bounded self-adjoint Fredholm operators. By our spectral gap condition, this family defines an element  $\mathcal{I}_{\text{Corner}}$  of the  $K$ -group  $K_1(C(\mathbb{T})) \cong [\mathbb{T}, \mathcal{F}_*^{s,a}]$ . We call  $\mathcal{I}_{\text{Corner}}$  the *corner invariant*.

**3.3. Correspondence.** The following is the main theorem of this paper.

**Theorem 3.6.** *The map  $\delta_0: K_0(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T})) \rightarrow K_1(C(\mathbb{T}))$  maps the bulk-edge invariant to the corner invariant, that is,  $\delta_0(\mathcal{I}_{\text{Bulk-Edge}}) = \mathcal{I}_{\text{Corner}}$ .*

*Proof.* Let  $q := (1-h)(H^\alpha(t) - \mu, H^\beta(t) - \mu)$ . Since  $\delta_0([p]_0 + [q]_0) = \delta_0[1_N]_0 = 0$ , we have  $\delta_0(\mathcal{I}_{\text{Bulk-Edge}}) = -\delta_0[q]_0 = [\exp(2\pi i \hat{q})]_1$ , where  $\hat{q} := \rho(q)$  is a self-adjoint lift of  $q$ . By our spectral gap condition and Theorem 2.1, we have  $\text{ess-sp}(\hat{q}(t)) = \text{sp}(q(t)) = \{0, 1\}$ . By considering a spectral deformation which collapses eigenvalues in some small neighborhoods of 0 and 1 to points 0 and 1, respectively, we can deform  $\hat{q}(t)$  into an element  $\tilde{q}(t)$  of  $\hat{F}_*^\infty$ . Thus we obtain  $[\exp(2\pi i \hat{q})]_1 = [\exp(2\pi i \tilde{q})]_1$ . Let us consider the isomorphisms  $K_1(C(\mathbb{T})) = [\mathbb{T}, \mathbf{U}(\infty)] \cong [\mathbb{T}, \hat{F}_*^\infty] \cong [\mathbb{T}, \mathcal{F}_*^{s,a}]$ . Then we have the following relation.

$$\delta_0(\mathcal{I}_{\text{Bulk-Edge}}) = [\exp(2\pi i \tilde{q})]_1 = [2\tilde{q} - 1] = [2\tilde{q} - 1].$$

Since two loops  $\{2\tilde{q}(t) - 1\}_{t \in \mathbb{T}}$  and  $\{H^{\alpha,\beta}(t) - \mu\}_{t \in \mathbb{T}}$  are homotopic in  $\mathcal{F}_*^{s,a}$ , we have  $[2\tilde{q} - 1] = [\{H^{\alpha,\beta}(t) - \mu\}_{t \in \mathbb{T}}] = \mathcal{I}_{\text{Corner}}$ .  $\square$

<sup>7</sup>In order to define the element  $[p]_0$  of the  $K$ -group  $K_0(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T}))$ , we took the orthonormal frame of  $V$ . However, this element does not depend on the choice.



Remark 3.7. By the diagram (2), the map  $\text{sf}: K_1(C(\mathbb{T})) \rightarrow \mathbb{Z}$  maps our invariants to the spectral flow of  $\{H^{\alpha,\beta}(t) - \mu\}_{t \in \mathbb{T}}$ . The spectral flow of  $\{H^{\alpha,\beta}(t) - \mu\}_{t \in \mathbb{T}}$  is defined by counting wave functions localized near the corner. Thus, if the corner invariant is non-trivial, there exists topologically protected *corner states*. Note that the bulk-edge invariant cannot change unless the spectral gap of edges closes. By Theorem 3.6, this stability also holds for corner invariants.

Remark 3.8. Since the map  $\delta_0: K_0(\mathcal{S}^{\alpha,\beta} \otimes C(\mathbb{T})) \rightarrow K_1(C(\mathbb{T}))$  is not an isomorphism (Remark 2.4), bulk-edge invariants may have more information than corner invariants.

Remark 3.9. If the spectrum of  $(H^\alpha(t) - \mu, H^\beta(t) - \mu)$  is contained in  $\mathbb{R}_{>0}$  or  $\mathbb{R}_{<0}$ , then we can define the bulk-edge invariant in the same way. The spectral flow of the family  $\{H^{\alpha,\beta}(t) - \mu\}_{t \in \mathbb{T}}$  is also defined.<sup>8</sup> In this case, it is easy to see that  $\delta_0(\mathcal{I}_{\text{Bulk-Edge}}) = 0$  (see the proof of Theorem 3.6) and  $\text{sf}(\{H^{\alpha,\beta}(t) - \mu\}_{t \in \mathbb{T}}) = 0$ . Thus we still have such “bulk-edge and corner” correspondence.

Remark 3.10. Since  $\mathbb{T}$  is just a parameter space in our formulation, we can generalize the parameter space  $\mathbb{T}$  to other spaces. Let  $X$  be a compact Hausdorff space, and consider a continuous family of bounded self-adjoint multiplication operators generated by continuous functions  $H: X \rightarrow B(L^2(\mathbb{T} \times \mathbb{T}; V))$ . Then we have edge Hamiltonians and the corner Hamiltonian. If we assume our spectral gap condition, we can define the bulk-edge invariant and the corner invariant in the same way as elements of  $K_0(\mathcal{S}^{\alpha,\beta} \otimes C(X))$  and  $K_1(C(X))$ , respectively. As in Theorem 3.6, it is easily checked that the bulk-edge invariant maps to the corner invariant by the exponential map  $\delta_0$ . In this case we lack the understanding of the corner invariant as the spectral flow, but we still have a relation with our invariants and corner states. It is easily checked that if the corner invariant is non-trivial, then there are topologically protected corner states. In particular, if we take  $X = \mathbb{T} \times \mathbb{T}$ , then this argument gives a “bulk-edge and corner” correspondence for such four dimensional systems.

## REFERENCES

- [APS75] Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry. I. Math. Proc. Cambridge Philos. Soc., **77**, 43–69, (1975)
- [AS69] Atiyah, M.F., Singer, I.M.: Index theory for skew-adjoint Fredholm operators. Inst. Hautes Études Sci. Publ. Math. **37**, 5–26 (1969)
- [ASBVB13] Avila, J.C., Schulz-Baldes, H., Villegas-Blas, C.: Topological invariants of edge states for periodic two-dimensional models. Math. Phys. Anal. Geom. **16**(2), 137–170 (2013)
- [Bel86] Bellissard, J.: *K*-theory of  $C^*$ -algebras in solid state physics. Statistical mechanics and field theory: mathematical aspects (Groningen, 1985), pp. 99–156, Springer, Berlin (1986)
- [BvESB94] Bellissard, J., van Elst, A., Schulz-Baldes, H.: The noncommutative geometry of the quantum Hall effect. J. Math. Phys. **35**(10), 5373–5451 (1994)
- [Bla98] Blackadar, B.: *K-theory for operator algebras*: Second edition. Mathematical Sciences Research Institute Publications, vol 5. Cambridge University Press, Cambridge, (1998)
- [BS06] Böttcher, A., Silbermann, B.: *Analysis of Toeplitz operators*: Second edition. Springer-Verlag, Berlin (2006)
- [BCE15] Bourne, C., Carey, A.L., Rennie, A.: The bulk-edge correspondence for the quantum Hall effect in Kasparov theory. Lett. Math. Phys. **105**(9), 1253–1273 (2015)

<sup>8</sup>In this case, the family  $\{H^{\alpha,\beta}(t) - \mu\}_{t \in \mathbb{T}}$  is not contained in  $\mathcal{F}_*^{s,a}$ , and does not define an element of the  $K$ -group  $K_1(C(\mathbb{T}))$ .

- [BKR17] Bourne, C., Kellendonk, J., Rennie, A.: The  $K$ -theoretic bulk-edge correspondence for topological insulators. *Ann. Henri Poincaré* (2017). doi:10.1007/s00023-016-0541-2
- [CD71] Coburn, L.A., Douglas, R.G.:  $C^*$ -algebras of operators on a half-space. I. *Inst. Hautes Études Sci. Publ. Math.* **40**, 59–67 (1971)
- [CDS72] Coburn, L.A., Douglas, R.G., Singer, I.M.: An index theorem for Wiener-Hopf operators on the discrete quarter-plane. *J. Differ. Geom.* **6**, 587–593 (1972)
- [CDSS71] Coburn, L.A., Douglas, R.G., Schaeffer, D.G., Singer, I.M.:  $C^*$ -algebras of operators on a half-space. II. Index theory. *Inst. Hautes Études Sci. Publ. Math.* **40**, 69–79 (1971)
- [DH71] Douglas, R.G., Howe, R.: On the  $C^*$ -algebra of Toeplitz operators on the quarter-plane. *Trans. Amer. Math. Soc.* **158**, 203–217 (1971)
- [Dou98] Douglas, R.G.: *Banach algebra techniques in operator theory*: Second edition. volume 179 of Graduate Texts in Mathematics. Springer-Verlag, New York (1998)
- [EG02] Elbau, P., Graf, G.M.: Equality of bulk and edge Hall conductance revisited. *Comm. Math. Phys.* **229**(3), 415–432 (2002)
- [GP13] Graf, G.M., Porta, M.: Bulk-edge correspondence for two-dimensional topological insulators. *Comm. Math. Phys.* **324**(3), 851–895 (2013)
- [Hat93a] Hatsugai, Y.: Edge states in the integer quantum Hall effect and the Riemann surface of the Bloch function. *Phys. Rev. B* **48**, 11851–11862 (1993)
- [Hat93b] Hatsugai, Y.: Chern number and edge states in the integer quantum Hall effect. *Phys. Rev. Lett.* **71**, 3697–3700 (1993)
- [Hay16] Hayashi, S.: Bulk-edge correspondence and the cobordism invariance of the index. preprint, [arXiv:1611.08073 \[math-ph\]](https://arxiv.org/abs/1611.08073), (2016)
- [HR00] Higson, N., Roe, J.: *Analytic K-homology*. Oxford Mathematical Monographs. Oxford University Press, Oxford Science Publications, Oxford (2000)
- [Jia95] Jiang, X.: On Fredholm operators in quarter-plane Toeplitz algebras. *Proc. Amer. Math. Soc.* **123**(9), 2823–2830 (1995)
- [KR SB02] Kellendonk, J., Richter, T., Schulz-Baldes, H.: Edge current channels and Chern numbers in the integer quantum Hall effect. *Rev. Math. Phys.* **14**(1), 87–119 (2002)
- [Koh85] Kohmoto, M.: Topological invariant and the quantization of the hall conductance. *Annals of Physics*, **160**, 343–354 (1985)
- [Kub16] Kubota, Y.: Controlled topological phases and bulk-edge correspondence. *Comm. Math. Phys.* (2016). doi:10.1007/s00220-016-2699-3
- [MT16] Mathai, V., Thiang, G.C.: T-duality simplifies bulk-boundary correspondence. *Comm. Math. Phys.* **345**, 675 (2016). doi:10.1007/s00220-016-2619-6
- [Mur90] Murphy, G.J.:  *$C^*$ -algebras and operator theory*. Academic Press, Inc., Boston, MA, (1990)
- [Par90] Park, E.: Index theory and Toeplitz algebras on certain cones in  $\mathbb{Z}^2$ . *J. Operator Theory*, **23**(1), 125–146, (1990)
- [PS91] Park, E., Schochet, C.: On the  $K$ -theory of quarter-plane Toeplitz algebras. *Internat. J. Math.* **2**(2), 195–204, (1991)
- [Phi96] Phillips, J.: Self-adjoint Fredholm operators and spectral flow. *Canad. Math. Bull.* **39**(4), 460–467 (1996)
- [PSB16] Prodan, E., Schulz-Baldes, H.: *Bulk and boundary invariants for complex topological insulators*: From  $K$ -theory to physics. Springer, Springer International Publishing, Berlin (2016)
- [RLL00] Rørdam, M., Larsen, F., Laustsen, N.: *An introduction to  $K$ -theory for  $C^*$ -algebras*. volume 49 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, (2000)
- [SBKR00] Schulz-Baldes, H., Kellendonk, J., Richter, T.: Simultaneous quantization of edge and bulk Hall conductivity. *J. Phys. A* **33**(2), L27–L32 (2000)
- [TKNdN82] Thouless, D.J., Kohmoto, M., Nightingale, M.P., den Nijs, M.: Quantized Hall conductance in a two-dimensional periodic potential. *Phys. Rev. Lett.* **49**, 405–408 (1982)
- [WO93] Wegge-Olsen, N.E.:  *$K$ -theory and  $C^*$ -algebras*. A friendly approach. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, (1993).

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, 3-8-1 KOMABA,  
TOKYO, 153-8914, JAPAN.

*E-mail address:* hayashi@ms.u-tokyo.ac.jp